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Heteroscedasticity in a two-factors design model

Heteroscedasticidad en un modelo de diseño de dos factores

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ABSTRACT

This paper studies the two-factors design model when the heteroscedasticity of variance is present in errors. As can be observed, testing of hypothesis based on the main effects for this design model can be performed using Hotelling's T^2 test. Simultaneous confidence intervals are also proposed. Finally, the proposed methodology is applied to a real-life example.

RESUMEN

En el presente trabajo se estudia el modelo de diseño de dos factores cuando se presenta la heteroscedasticidad de varianza en los errores. Como se verá, la prueba de hipótesis sobre los efectos principales para tal modelo de diseño se puede realizar a través del estadístico T^2 de Hotelling. Además son propuestos intervalos de confianza simultáneos. Finalmente la metodología propuesta es aplicada a un ejemplo real.

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Palabras clave:
Modelo de diseño de dos factores; heterocedasticidad de varianza; estructura de covarianza.

Keywords:
Two-factors design model; heteroscedasticity of variance; covariance structure.

INTRODUCTION

Consider the **two-factors design model**

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \quad i = 1,2,\ldots,t; \quad j = 1,2,\ldots,r. \tag{1}$$

where y_{ij} are observable random variables. The ε_{ij} are unobservable random variables, they are independent and are both normally distributed, furthermore $\varepsilon_{ij} \sim \mathcal{N}(0,\sigma^2)$.

$\sigma^2, \mu, \tau_1, \tau_2, \ldots, \tau_t, \beta_1, \beta_2, \ldots, \beta_r$ are unknown parameters, and the parameter space is $\mathbf{\Omega}$, where

$$\mathbf{\Omega} = \left\{\left(\sigma^2, \mu, \tau_1, \tau_2, \ldots, \tau_t, \beta_1, \beta_2, \ldots, \beta_r\right) \mid \sigma^2 > 0, -\infty < \mu < \infty, -\infty < \tau_i < \infty, i = 1,2,\ldots,t, -\infty < \beta_j < \infty, j = 1,2,\ldots,r\right\}.$$

Consider that there is interest in testing the hypothesis:

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_t. \tag{2}$$

Remark

Let $T_1, T_2, \ldots, T_t$ be the t treatments and let τ_i, $i = 1,2,\ldots,t$ be the corresponding associated parameters to each treatment T_i. The **main effect** of the ith treatment is defined as $\tau_j - \overline{\tau}$, where $\overline{\tau} = t^{-1}\sum_{j=1}^{t}\tau_i$. Thus, formally, the hypothesis of interest in this model is that: the main effects of all t treatments are equal. That is:

$$H_0 : \tau_1 - \overline{\tau} = \tau_2 - \overline{\tau} = \cdots = \tau_t - \overline{\tau},$$

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which is equivalent to

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_t. \tag{3}$$

And the alternative hypothesis H_a is: at least one equality is an inequality. In addition remember that the mean of the treatment T_i is $\mu + \tau_i$, $i = 1,2,...,t$.

In this paper it is assumed that certain assumptions about the model (1) are not met. Thereby, alternatively it is assumed that errors ε_{ij} satisfy the following conditions:

$$\begin{aligned} E(\varepsilon_{ij}^2) &= \sigma_{ii} \\ E(\varepsilon_{ij},\varepsilon_{kl}) &= \sigma_{ik} \quad \text{si } j = l \\ E(\varepsilon_{ij},\varepsilon_{kl}) &= 0 \quad \text{si } j \neq l. \end{aligned} \tag{4}$$

These premises on ε_{ij}, establish that the observations are uncorrelated if they are in different blocks; that the variance of the ith treatment observation σ_{ii}; and that the covariance of the ith treatment observation and kth treatment observation in the same block σ_{ik}. In some instances these premises seem to be more realistic than those generally made, i.e., that the ε_{ij} are distributed independently $\mathcal{N}(0,\sigma^2)$

Furthermore, it is assumed that the errors are not normally independent and identically distributed, instead it is assumed that errors ε_{ij} have an elliptical joint distribution with covariance structure such that the set of specifications stated by (4) are satisfied, which means that errors are not independent but, perhaps, only uncorrelated. In literature, this problem was addressed by Graybill (1961) under the same assumption of heteroscedasticity of variance, but assuming normality.

The work is presented as follows: the first part gathers some results of matrix algebra and multivariate statistics, while introducing the notation that will be used. The main contribution of this work is developed in advanced, where the methodology used to test the hypothesis (2) is proposed, under the covariance structure specified by (4), i.e. under heteroscedasticity of variance for the two-factors design model (1). The article, concludes with the application of the proposed methodology to a real-life example.

Preliminary results

A comprehensive discussion of matrix algebra and multivariate statistical analysis can be found in Harville (2008) and Muirhead (1982). For convenience, some notations will be introduced, although in general the authors have adhered to standard notations.

If $\mathbf{A}$ is $n \times m$ matrix, $\mathbf{A}'$: $n \times m$ denotes the **transpose** of $\mathbf{A}$. Generically, if $\mathbf{A}$ is $n \times m$, it shall be written in term of their elements, rows or columns, respectively, as

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} \mathbf{a}'_{(1)} \\ \mathbf{a}'_{(2)} \\ \vdots \\ \mathbf{a}'_{(n)} \end{bmatrix} = [\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_m],$$

where $\mathbf{a}_{(i)}$ is a vector $m \times 1$, i = 1,2...,n and $\mathbf{a}_j$ is a vector $n \times 1$, j = 1,2...,m. If $\mathbf{A}$ is a square matrix of order n it is termed **symmetric** if $\mathbf{A} = \mathbf{A}'$. The **identity matrix** of order n is denoted by $\mathbf{I}_n$. The vector with ones in each position of order n is denotes as $\mathbf{1}_n = (\underset{1}{1}, \underset{2}{1}, \underset{\cdots}{\ldots}, \underset{n}{1})'$ and the kth **vector of the canonical base** of order n is denoted as $\mathbf{e}_k^n = (\underset{1}{0}, \underset{2}{0}, \underset{\cdots}{\ldots}, \underset{k-1}{0}, \underset{k}{1}, \underset{k+1}{0}, \underset{\cdots}{\ldots}, \underset{n}{0})'$. Similarly, **vector with zeros** in each position of order n is denotes as $\mathbf{0}_n = (\underset{1}{0}, \underset{2}{0}, \underset{\cdots}{\ldots}, \underset{n}{0})'$.

▫ Definition 1

If $\mathbf{A}$ is a $n \times m$ matrix then by vec($\mathbf{A}$) that is the $mn \times 1$ vector formed by stacking the columns of $\mathbf{A}$ under each other; that is, if

$\mathbf{A} = (\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_m)$

where $\mathbf{a}_j$ is $n \times 1$, $j = 1,2,...,m$, then

$$\text{vec}(\mathbf{A}) = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}.$$

▫ Definition 2

Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix and $\mathbf{B} = (b_{ij})$ be a $p \times q$ matrix. The **Kronecker product** (also known as **direct product** or **tensor product**) of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is the $mp \times nq$ matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

The relation between Kronecker product and the of a matrix is specified in the following lemma.

▫ Lemma 1

If $\mathbf{B}$ is $r \times m$, $\mathbf{Y}$ is $m \times n$, and $\mathbf{C}$ is $n \times s$ then

$$\text{vec}(\mathbf{BXC}) = (\mathbf{C}' \otimes \mathbf{B})\text{vec}(\mathbf{Y})$$

Now, **the generalized multivariate elliptical matrix distributions** are introduced in this section. A comprehensive and systematic study can be found in Fang & Zhang (1990) and Gupta & Varga (1993).

▫ Definition 3

It is said that the random matrix **Y**: $n \times m$ has a *variate elliptical matrix distribution,* denoted as $\mathbf{Y} \sim \mathcal{E}_{n\times m}(\boldsymbol{\mu}, \boldsymbol{\Theta}, \boldsymbol{\Sigma}, h)$, if its density with respect to the Lebesgue measure is

$$\frac{C(m,n)}{|\boldsymbol{\Sigma}|^{n/2}|\boldsymbol{\Theta}|^{m/2}} h\left\{\mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu})'\boldsymbol{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right]\right\}. \quad (5)$$

where

$$C(m,n) = \frac{\Gamma[mn/2]}{2\pi^{mn/2}}\left\{\int_{u>0} u^{mn-1}h(u^2)du\right\}^{-1} \quad (6)$$

and $\boldsymbol{\Theta}$ is $n \times m$, $\boldsymbol{\Sigma}$ is $m \times m$ and $\boldsymbol{\mu}$ is $n \times m$ are constant matrices, such that $\boldsymbol{\Theta}$ and $\boldsymbol{\Sigma}$ are symmetric positive definite matrices. Also, in (5), tr(·) denotes the trace, $|\mathbf{A}|$ denotes the determinant of **A** and in (6), Γ[] denotes the gamma function.

Observe that this class of multivariate matrix distributions includes normal, contaminated normal, Pearson type II and VII, Kotz, Jensen-Logistic, power exponential and Bessel distributions, among others; these distributions have tails that are more or less weighted, and/or present a greater or smaller degree of kurtosis than the matrix multivariate normal distribution. In particular, observe that if in Definition 3 h is taken that $h(u) = \exp(-u/2)$, from (6) it can be readily seen that $C(m,n) = (2\pi)^{-mn/2}$. Hence, the density obtained is

$$\frac{1}{(2\pi)^{mn/2}|\boldsymbol{\Sigma}|^{n/2}|\boldsymbol{\Theta}|^{m/2}}\,\mathrm{etr}\left\{-\frac{1}{2}\left[\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu})'\boldsymbol{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right]\right\}, \quad (7)$$

which is named, the **multivariate matrix normal distribution** and is denoted as $\mathbf{Y} \sim \mathcal{N}_{n\times m}(\boldsymbol{\mu}, \boldsymbol{\Theta}, \boldsymbol{\Sigma})$. In (7), $\mathrm{etr}\{\} \equiv \exp\{\mathrm{tr}()\}$.

Similarly, observe that if in Definition 3 h is taken as $h(u) = (1+u/\upsilon)^{-s}$, where $s, \nu \in \Re$, $s, \upsilon > 0$, $s > mn/2$; from (6) it can be seen that

$$C(m,n) = \frac{\Gamma[s]}{(\pi\nu)^{mn/2}\Gamma\left[s - \frac{mn}{2}\right]}.$$

Therefore, the density is

$$\frac{(\pi\nu)^{-mn/2}\Gamma[s]}{\Gamma\left[s-\frac{mn}{2}\right]|\boldsymbol{\Sigma}|^{n/2}|\boldsymbol{\Theta}|^{m/2}}\left\{1+\frac{1}{\nu}\mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu})'\boldsymbol{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right]\right\}^{-s}, \quad (8)$$

which is termed the **multivariate matrix Pearson type VII distribution**. Observe that when $s = (mn+\upsilon)/2$ in (8), **Y** is said to have a **multivariate matrix *t* distribution** with υ degrees of freedom. And in this case, if $\upsilon = 1$, then **Y** is said to have a **multivariate matrix Cauchy distribution.**

The following result summarises some basic properties of elliptical distributions.

▫ Proposition 1

Let $\mathbf{Y} \sim \mathcal{E}_{n\times m}(\boldsymbol{\mu}, \boldsymbol{\Theta}, \boldsymbol{\Sigma}, h)$.

1. *Then the characteristic function of* **Y** *is*

$$\Phi_{\mathbf{Y}}(\mathbf{T}) = \mathrm{etr}\{i\mathbf{T}'\boldsymbol{\mu}\}\psi(\mathrm{tr}(\mathbf{T}'\boldsymbol{\Theta}\mathbf{T}\boldsymbol{\Sigma}))$$

2. *Assume* **C** *is* $p \times q$, **A** *is* $p \times n$ *and* **B** *is* $m \times q$ *are constant matrices. Then*

$$\mathbf{AYB} + \mathbf{C} \sim \mathcal{E}_{p\times q}(\mathbf{A}\boldsymbol{\mu}\mathbf{B} + \mathbf{C}, \mathbf{A}\boldsymbol{\Theta}\mathbf{A}', \mathbf{B}'\boldsymbol{\Sigma}\mathbf{B}, h)$$

3. *If* **Y** *has a finite first and second moments, then*

(a) $E(\mathbf{Y}) = \mu$,

(b) $\mathrm{Cov}(\mathrm{vec}\mathbf{Y}) = c\boldsymbol{\Theta} \otimes \boldsymbol{\Sigma}$, and

(c) $\mathrm{Cov}(\mathrm{vec}\mathbf{Y}') = c\boldsymbol{\Sigma} \otimes \boldsymbol{\Theta}$,

where $c = -2\psi'(0)$.

In particular when $\mathbf{Y} \sim \mathcal{N}_{n\times m}(\boldsymbol{\mu}, \boldsymbol{\Theta}, \boldsymbol{\Sigma})$ in Proposition 1, then $c = 1$.

Let $\mathbf{Y} \sim \mathcal{E}_{n\times m}(\mathbf{1}_n\boldsymbol{\mu}', \mathbf{I}_n, \boldsymbol{\Sigma}, h)$ with $n > m$ and let h be non-increasing and continuous with $\boldsymbol{\mu} : m \times 1$ and $\boldsymbol{\Sigma} : m \times m$ are unknown. It is interested in whether $\boldsymbol{\mu}$ equals a specific $\boldsymbol{\mu}_0$, that is, it is required to test the hypothesis

$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$, against $H_a : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$,

Without loss of generality $\boldsymbol{\mu}_0 = \mathbf{0}$, is assumed, otherwise, it may consider to replace **Y** by $\mathbf{Y} - 1_n\boldsymbol{\mu}_0'$. Therefore the above hypothesis becomes

$$H_0 : \boldsymbol{\mu} = \mathbf{0}, \text{ against } H_a : \boldsymbol{\mu} \neq \mathbf{0}. \quad (9)$$

By Fang & Zhang (1990) (see also Gupta & Varga, 1993) it easy verify that under null hypothesis (9) the corresponding statistic test is **invariant regarding the family of elliptical distributions** (5). Then to determine the statistic test and its null distribution it is sufficient to study the latter under normality.

From Fang & Zhang (1990) and Muirhead (1982) under likelihood ratio criteria or from Srivastava & Khatri (1979) under the **union-intersection principle** of test construction of Roy, the rejection region for a test of level α is

$$T^2 > \frac{m(n-1)}{n-m}F_{\alpha,m,n-m} \quad (10)$$

where $F_{\alpha,m,n-m}$ denotes the 100α upper percentage point of the F distribution, with m, $n-m$ degrees of freedom and T^2 the **Hotelling statistic** is defined by

$$T^2 = n\bar{\mathbf{Y}}'\mathbf{S}^{-1}\bar{\mathbf{Y}},$$

with

$$\bar{\mathbf{Y}} = \frac{1}{n}\mathbf{Y}'\mathbf{1}_n \ \ and\ \ \mathbf{S} = \frac{1}{n-1}\mathbf{Y}'\left(\mathbf{I}_n - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)\mathbf{Y}.$$

The significance of T^2 still leaves the question of which particular equally $\mu_j = \mu_{0j}$, $j = 1,2,\ldots,m$ (in $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$) unaswered which have probably lead to the rejection of the vector hypothesis. While it might help test the individual hypothesis by referring their univariate t statistics to the Bonferroni critical values, the union-intersection nature of the T^2 test leads directly to a way of controlling the Type I error probability for the tests on all linear functions of the response means $\mathbf{a}'\boldsymbol{\mu}$, where is any nonnull vector $m \times 1$. Thus, the family of **simultaneous confidence intervals of Roy and Box** with coefficient $1-\alpha$ for all choices of the elements of $\mathbf{a}$ in $\mathbf{a}'\boldsymbol{\mu}$ are

$$\mathbf{a}'\bar{\mathbf{Y}} - T_{\alpha,m,n-m}\sqrt{\frac{1}{n}\mathbf{a}'\mathbf{S}\mathbf{a}} \leq \ \mathbf{a}'\boldsymbol{\mu} \leq \ \mathbf{a}'\bar{\mathbf{Y}} - \ T_{\alpha,m,n-m}\sqrt{\frac{1}{n}\mathbf{a}'\mathbf{S}\mathbf{a}}$$

where

$$T^2_{\alpha,m,n-m} = \frac{m(n-1)}{n-m}F_{\alpha,m,n-m}.$$

PROPOSED METHODOLOGY

First observe that, alternatively, the model (1) can be re-written as

$$\mathbf{y} = \mathbf{X}\mathfrak{B} + \varepsilon$$

where

$$\mathbf{y} = (y_{11}, y_{12}, \ldots, y_{1r}, y_{21}, y_{22}, \ldots, y_{2r}, \ldots, y_{t1}, y_{t2}, \ldots, y_{tr})',$$

$$= (\mathbf{y}'_1, \mathbf{y}'_2, \ldots \mathbf{y}'_t)', \tag{11}$$

$\mathbf{y}_i = (y_{i1}, y_{i2}, \ldots, y_{ir})'$, $i = 1,2,\ldots,t$, and

$$\mathbf{X} = (\mathbf{1}_{tr} \mid \mathbf{I}_t \otimes \mathbf{1}_r \mid \mathbf{1}_t \otimes \mathbf{I}_r), \quad \mathfrak{B} = \begin{bmatrix}\mu\\ \tau\\ \beta\end{bmatrix}, \tau = \begin{bmatrix}\tau_1\\ \tau_2\\ \vdots\\ \tau_r\end{bmatrix}, \beta = \begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_r\end{bmatrix},$$

with

$$\varepsilon = (\varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{1r}, \varepsilon_{21}, \varepsilon_{22}, \ldots, \varepsilon_{2r}, \ldots, \varepsilon_{t1}, \varepsilon_{t2}, \ldots, \varepsilon_{tr})',$$

such that,

$$\varepsilon \sim \mathcal{N}_{tr}(\mathbf{0}, \sigma^2\mathbf{I}_{tr}). \tag{12}$$

Now, note that under the assumptions (4), the assumption (12) is modified to

$$\varepsilon \sim \mathcal{N}_{tr}(\mathbf{0}, \boldsymbol{\Sigma} \otimes \mathbf{I}_r), \tag{13}$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1t}\\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{1t}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{t1} & \sigma_{t2} & \cdots & \sigma_{tt}\end{bmatrix}$$

Under this assumption it is interesting to test the hypothesis (3).

With this aim in mind, the vector (11) $\mathbf{y}$, can be rewritten as the matrix

$$\mathbb{Y} = [\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_t] = \begin{bmatrix}y_{11} & y_{21} & \cdots & y_{t1}\\ y_{12} & y_{22} & \cdots & y_{t2}\\ \vdots & \vdots & \ddots & \vdots\\ y_{1r} & y_{2r} & \cdots & y_{tr}\end{bmatrix}$$

and note that $\mathbf{y} = \operatorname{vec}(\mathbb{Y})$. Then

$$\mathbb{Y} \sim \mathcal{N}_{r\times t}(\boldsymbol{\Gamma}, \mathbf{I}_r, \boldsymbol{\Sigma}),$$

where

$$\boldsymbol{\Gamma} = \begin{bmatrix}\mu+\tau_1+\beta_1 & \mu+\tau_2+\beta_1 & \cdots & \mu+\tau_t+\beta_1\\ \mu+\tau_1+\beta_2 & \mu+\tau_2+\beta_2 & \cdots & \mu+\tau_t+\beta_2\\ \vdots & \vdots & \ddots & \vdots\\ \mu+\tau_1+\beta_r & \mu+\tau_2+\beta_{r2} & \cdots & \mu+\tau_t+\beta_r\end{bmatrix}$$

Now, let

$$\mathbb{Y}_1 = \mathbb{Y}\mathbf{M} = [\mathbf{y}_2 \cdots \mathbf{y}_t]$$

where $\mathbf{M} = [\mathbf{0}_{t-1}\mathbf{I}_{t-1}]'$. Also, note that $\mathbf{y}_1 = \mathbb{Y}\mathbf{e}_1^t$. Then, define

$$\mathbb{Y}_2 = \mathbb{Y}_1 - \mathbf{y}_1\mathbf{1}'_{t-1}$$

$$= \begin{bmatrix}y_{21}-y_{11} & y_{31}-y_{11} & \cdots & y_{t1}-y_{11}\\ y_{22}-y_{12} & y_{32}-y_{12} & \cdots & y_{t2}-y_{12}\\ \vdots & \vdots & \ddots & \vdots\\ y_{2r}-y_{1r} & y_{3r}-y_{1r} & \cdots & y_{tr}-y_{1r}\end{bmatrix}$$

$$= \mathbb{Y}\left(\mathbf{M} - \mathbf{e}_1^t\mathbf{1}'_{t-1}\right),$$

and observe that

$$\mathrm{E}(\mathbb{Y}_2) = \mathrm{E}(\mathbb{Y})\left(\mathbf{M} - \mathbf{e}_1^t\mathbf{1}'_{t-1}\right)$$

$$= \boldsymbol{\Gamma}\left(\mathbf{M} - \mathbf{e}_1^t\mathbf{1}'_{t-1}\right)$$

$$= \begin{bmatrix}\tau_2-\tau_1 & \tau_3-\tau_1 & \cdots & \tau_t-\tau_1\\ \tau_2-\tau_1 & \tau_3-\tau_1 & \cdots & \tau_t-\tau_1\\ \vdots & \vdots & \ddots & \vdots\\ \tau_2-\tau_1 & \tau_3-\tau_1 & \cdots & \tau_t-\tau_1\end{bmatrix}$$

$$= \mathbf{1}_r\gamma'$$

where $\gamma = (\tau_2 - \tau_1, \tau_3 - \tau_1, \ldots, \tau_t - \tau_1)'$. And

$$\text{Cov}(\text{vec}(\mathbb{Y}_2)) = \text{Cov}\left(\text{vec}\left(\mathbb{Y}\left(\mathbf{M} - \mathbf{e}_1^t \mathbf{1}'_{t-1}\right)\right)\right)$$

$$= \text{Cov}\left(\left(\left(\mathbf{M} - \mathbf{1}_{t-1}\mathbf{e}_1^{t\prime}\right) \otimes \mathbf{I}_r\right)\text{vec}(\mathbb{Y})\right)$$

$$= \left(\mathbf{M} - \mathbf{e}_1^t \mathbf{1}'_{t-1}\right)\mathbf{\Sigma}\left(\mathbf{M}' - \mathbf{1}_{t-1}\mathbf{e}_1^{t\prime}\right) \otimes \mathbf{I}_r.$$

Hence

$$\mathbb{Y}_2 \sim \mathcal{N}_{r\times(t-1)}\left(\mathbf{1}_r\gamma', \mathbf{I}_r, \left(\mathbf{M} - \mathbf{e}_1^t\mathbf{1}'_{t-1}\right)\mathbf{\Sigma}\left(\mathbf{M}' - \mathbf{1}_{t-1}\mathbf{e}_1^{t\prime}\right)\right)$$

Then observe that $\tau_1 = \tau_2 = \cdots = \tau_t$ if and only if $\boldsymbol{\gamma} = \mathbf{0}$. Therefore if $r > t$ and

$$\overline{\mathbb{Y}}_2 = \frac{1}{r}\mathbb{Y}'_2\mathbf{1}'_r \;\; and \;\; \mathbb{S} = \frac{1}{r-1}\mathbb{Y}'_2\left(\mathbf{I}_r - \frac{1}{r}\mathbf{1}_r\mathbf{1}'_r\right)\mathbb{Y}_2,$$

by (9) and (10) the following decision rule is obtained

$$\text{Reject } H_0 : \tau_1 = \tau_2 = \cdots = \tau_t \;\text{ if }\; T^2 > \frac{(t-1)(r-1)}{(r-t+1)}F_{\alpha,t-1,r-t+1},$$

where $T^2 = r\overline{\mathbb{Y}}_2{}'\mathbb{S}^{-1}\overline{\mathbb{Y}}_2$ and $r + 1 > t$.

In addition, the 100(1-α) percent simultaneous confidence bounds on all linear function $\mathbf{a}'\gamma$ are given by

$$\mathbf{a}'\overline{\mathbb{Y}}_2 - T_{\alpha,t-1,r-t+1}\sqrt{\frac{1}{b}\mathbf{a}'\mathbf{Sa}} \le \mathbf{a}'\gamma \le \mathbf{a}'\overline{\mathbb{Y}}_2 + T_{\alpha,t-1,r-t+1}\sqrt{\frac{1}{b}\mathbf{a}'\mathbf{Sa}}$$

where

$$T^2_{\alpha,t-1,r-t+1} = \frac{(t-1)(r-1)}{r-t+1}F_{\alpha,t-1,r-t+1}.$$

Observe that if instead of the assumption (13) it is assumed that

$$\varepsilon \sim \mathcal{N}_{tr}(\mathbf{0}, \mathbf{I}_t \otimes \mathbf{\Theta}), \tag{14}$$

where

$$\mathbf{\Theta} = \begin{bmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1r} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{r1} & \theta_{r2} & \cdots & \theta_{rr} \end{bmatrix}$$

Proceeding similarly it is possible to propose an analogous test for the hypothesis $H_0 : \beta_1 = \beta_2 = \cdots = \beta_r$.

It is emphasised that both the decision rule as well as simultaneous confidence intervals under the null hypothesis are invariant under the family of elliptical distributions, i.e. these results are in accordance with those obtained under the assumption of normality.

Example

An experiment was conducted in a randomised block design model $y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}$ where the assumptions of (4) hold. The data are shown in table 1.

Table 1.
Treatments

Block	1	2	3	4	5	6	7	8	9	10
1	30.5	23	15	24	22	17	34	20.5	30	14
2	20	18	25.5	18	33.5	14	14	26.5	13	16.5
3	24	14	24	14.5	19	24	30	32	20	2.5
4	14.5	14	18	20.5	15	24	29	15	31	20
5	28.5	28.5	27.5	15	34.5	27	17	29	26	15.5
6	16	14	13.5	18	27.5	25	8	7	13	6
7	34	20	15.5	16.5	20	21	6	21.5	4.5	19
8	37	17	21	17.5	15	26	17	26	24	22
9	19	19.5	17	17.5	20.5	18	21	24	34.5	17.5
10	37	5	18	24	17	19	14	30.5	2.5	21.5
11	27	15.5	29	18	14	32	7	13	4	25.5
12	27	28	31	16	15	18	2	10.5	28.5	14
13	14	42	15	22	17.5	8	17	4	17.5	11.5
14	23	29.5	16	20.5	26	10.5	17.5	28	2.5	2
15	18	23	29.4	28	30	11	4.5	27	5	20.5

It is interesting to test the hypothesis

$$H_0 : \tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = \tau_6 = \tau_7 = \tau_8 = \tau_9 = \tau_{10}. \tag{15}$$

By completeness, in table 2 the analysis of variance appears, ignoring that the assumptions (4) hold and in table 3 the analysis of variance is performed with the transformed data. The computations have been performed with the a support of program in language R. (Readers interested in obtaining a copy of this program, please contact the authors via email).

Table 2.
Analysis of Variance for the original data.

SV	DF	SS	MS	F	*p-value*
Blocks	14	888.58	63.47		
Treatments	9	1124.98	125	1.95	0.0507
Residuals	126	8078	64.11		
Total	149	10092			

Table 3.
Analysis of Variance for the transformed data (log(y_{ij})).

SV	DF	SS	MS	F	*p-value*
Blocks	14	4.99	0.3566		
Treatments	9	7.58	0.8417	2.634	0.00795
Residuals	126	40.27	0.3196		
Total	149	52.84			

From data in table 1 the following data are obtained

$$\overline{\mathbb{Y}}_2 = \begin{bmatrix} -3.900000 \\ -3.606667 \\ -5.300000 \\ -2.866667 \\ -5.000000 \\ -8.766667 \\ -3.666667 \\ -7.566667 \\ -9.433333 \end{bmatrix}$$

and $\mathbb{S}$ is

$$\begin{bmatrix} 181.50 & 78.85 & 87.94 & 106.25 & 30.33 & 81.17 & 23.76 & 111.66 & 44.76 \\ 78.85 & 89.86 & 56.84 & 75.08 & 46.72 & 38.52 & 35.74 & 68.90 & 51.17 \\ 87.94 & 56.84 & 83.38 & 79.57 & 31.58 & 71.09 & 34.14 & 65.37 & 52.12 \\ 106.25 & 75.08 & 79.57 & 134.66 & 42.92 & 72.53 & 64.09 & 75.08 & 41.86 \\ 30.33 & 46.72 & 31.58 & 42.92 & 69.92 & 48.60 & 10.89 & 67.46 & 37.71 \\ 81.17 & 38.52 & 71.09 & 72.53 & 48.60 & 164.38 & 57.73 & 138.92 & 28.98 \\ 23.76 & 35.74 & 34.14 & 64.09 & 10.89 & 57.73 & 78.91 & 28.68 & 11.53 \\ 111.66 & 68.90 & 65.37 & 75.08 & 67.46 & 138.92 & 28.68 & 221.17 & 53.89 \\ 44.76 & 51.17 & 52.12 & 41.86 & 37.71 & 28.98 & 11.53 & 53.89 & 73.46 \end{bmatrix}.$$

In which $T^2 = 31.9988$ with a *p-value* = 0.313276. This is significant at the 31.3 percent level; then, from an agronomic t traditional point of view, there is no evidence not to reject the hypothesis (15). Note that this conclusion is contrary to the conclusion achieved through the ANOVA tests, both with the original and transformed data.

As an example and even if the null hypothesis was not rejected, below the 100(1 - 0.05) percent simultaneous confidence intervals present in all comparisons between means $(\mu + \tau_i - (\mu + \tau_{i'}) = \tau_i - \tau_{i'} \quad i \neq i')$ are calculated (see table 4). The lower and upper limits of intervals in table 4, have been denoted as L_I and L_S, respectively.

By the properties of coherence and consonance of the union-intersection principle of Roy (see Gabriel, 1969), all intervals contain zero, meaning that all means are equal with a 100(1 - 0.05) percent simultaneous confidence coefficient, as expected.

Table 4.
Simultaneous confidence intervals, $\alpha = 0.05$.

		τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}
τ_1	L_I	-36.17	-26.32	-27.18	-30.67	-25.03	-39.48	-24.95	-43.19	-29.97
	L_S	28.37	19.10	16.58	24.93	15.03	21.95	17.61	28.06	11.10
τ_2	L_I		-25.83	-21.20	-25.42	-31.99	-27.59	-35.19	-28.41	-25.28
	L_S		25.25	24.00	23.36	34.19	37.32	34.72	35.75	36.35
τ_3	L_I			-16.80	-21.40	-18.12	-26.73	-23.57	-27.57	-12.88
	L_S			20.18	19.92	20.90	37.05	23.69	35.49	24.53
τ_4	L_I				-20.82	-23.04	-21.15	-24.86	-29.32	-13.24
	L_S				15.95	22.44	28.08	21.59	33.85	21.51
τ_5	L_I					-23.97	-23.83	-21.34	-29.66	-20.15
	L_S					28.24	35.63	22.94	39.06	33.28
τ_6	L_I						-24.28	-28.34	-27.37	-15.31
	L_S						31.82	25.67	32.50	24.18
τ_7	L_I							-32.18	-26.06	-31.46
	L_S							21.98	23.66	32.80
τ_8	L_I								-33.42	-21.47
	L_S								41.22	33.01
τ_9	L_I									-30.88
	L_S									34.61

CONCLUSIONS

Is important to emphasise that the proposed methodology is robust on all families of elliptical distributions; furthermore this can be extended to other design models. The trick is to find the corresponding matrix $\mathbb{Y}_2$ that allows testing the hypothesis of interest. In other texts the problem presented in this article has been usually solved through nonparametric tests or by applying one of the diverse transformations recommended on the original data, or a combination of them. See Montgomery (2005).

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