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Dynamics of the helmholtz oscillator with fractional order damping

Dinámicas del oscilador de hemholtz con amortiguación fraccionaria

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ABSTRACT

The dynamics of the nonlinear Helmholtz Oscillator with fractional order damping are studied in detail. The discretization of differential equations according to the Grünwald-Letnikov fractional derivative definition in order to get numerical simulations is reported. Comparison between solutions obtained through a fourth-order Runge-Kutta method and the fractional damping system are comparable when the fractional derivative of the damping term α is fixed at 1. That proves the good performance of the numerical scheme. The effect of taking the fractional derivative on the system dynamics is investigated using phase diagrams varying α from 0.5 to 1.75 with zero initial conditions. Periodic motions of the system are obtained at certain ranges of the damping term. On the other hand, escape of the trajectories from a potential well result at a certain critical value of the fractional derivative. The history of the displacement as a function of time is shown also for every α selected.

RESUMEN

Las dinámicas del Oscilador Helmholtz no-linear con amortiguación fraccionaria se estudian en detalle. Se reporta la discretización de ecuaciones diferenciales de acuerdo a la definición de derivada fraccionaria Grünwald-Letnikov para obtener simulaciones numéricas. Se comparan los resultados obtenidos para el caso no fraccionario mediante el método de Runge-Kutta de cuarto orden con el algoritmo fraccionario para el caso en el que el término fraccionario responsable del amortiguamiento del sistema α se fija en 1. Esto demuestra un buen rendimiento del esquema numérico. El efecto de tomar una derivada fraccionaria sobre las dinámicas del sistema se investiga al usar diagramas de fases variando α de 0.5 a 1.75 con condiciones iniciales de cero. Se obtienen movimientos periódicos del sistema en ciertos rangos determinados durante el amortiguamiento. Por el otro lado, el escape de las trayectorias de un pozo potencial resultará en un cierto valor crítico de la derivada fraccionaria.

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Helmholtz oscillator; Fractional damping; Grünwald-Letnikov fractional derivative.

Palabras clave: Oscilador Helmholtz; amortiguación fraccionaria; derivada fraccionaria Grünwald-Letnikov.

INTRODUCTION

Fractional differential and integral operators are mathematical tools very useful in engineering and scientific applications; one of these is control systems. The Helmholtz equation, which is being used in many physical problems, is modified to study the dynamics with a fractional order operator in the system. The rest of the parameters are kept constant during the work and the focus is on the effect of just taking fractional order operators in the damping term. Because this parameter plays an important role in the dynamics characteristics, it is necessary to study the impact of its variation in order to get information about fractional dynamics. In this work the Grünwald-Letnikov fractional derivative (Cao, Ma, Xie & Jiang, 2010; Monje, Chen, Vinagre, Xue & Feliu, 2010; Yang & Zhu, 2012) is considered for numerical simulations of the Helmholtz system.

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Fractional damped helmholtz system

The Helmholtz oscillator is a nonlinear second order differential equation (Almendral & Sanjuán, 2003; Almendral, Seoane & Sanjuán, 2004; Seoane, Zambrano, Euzzor, Meucci, Arecchi & Sanjuán, 2008). The motion of a particle with unit mass which undergoes a periodic forcing is given by

$$\ddot{x} + \mu\dot{x} + x - x^2 = Fcos(\omega t), \tag{1}$$

where μ, F and ω are positive constants. From the usual point of view, the damping term μ is proportional to the first order derivative of the displacement $x(t)$. In this work, the first order derivative $\dot{x}$ is replaced by a fractional derivative $D^{\alpha}x$, where α is the fractional damping exponent. The governing equation of the new system is

$$\ddot{x} + \mu D^{\alpha}x + x - x^2 = Fcos(\omega t). \tag{2}$$

The following property of fractional differential operators is very useful to get a new system with the purpose of numerical simulations.

$$D^{\alpha 1}D^{\alpha 2}x(t) = D^{\alpha 1+\alpha 2}x(t). \tag{3}$$

Discretization scheme

Equation (3) can be transformed into a new system with three fractional differential equations, which are given by

$$D^{\alpha}x = y, \tag{4.1}$$

$$D^{1-\alpha}y = z, \tag{4.2}$$

$$\frac{d}{dt}z = Dz = Fcos(wt) + x^2 - x - \mu y. \tag{4.3}$$

A numerical solution of the system above obtained by using Grünwald-Letnikov definition has the following form:

$$x(t_k) = y(t_{k-1})h^{\alpha} - \sum_{j=v}^{k} c_j^{(\alpha)}x(t_{k-j}), \tag{5.1}$$

$$y(t_k) = z(t_{k-1})h^{1-\alpha} - \sum_{j=v}^{k} c_j^{(1-\alpha)}y(t_{k-j}), \tag{5.2}$$

$$z(t_k) = [Fcos(wt_k) + x(t_k)^2 - \mu y(t_k) - x(t_k)]h - \sum_{j=v}^{k} c_j^{(1)}z(t_{k-j}), \tag{5.3}$$

where h is the discrete step. The coefficients c_j^{α} are the binomial coefficients derived of the Grünwald-Letnikov fractional derivative $c_0^{\alpha} = 1$ and

$$c_j^{\alpha} = \left(1 - \frac{\alpha + 1}{j}\right)c_{j-1}^{\alpha}. \tag{6}$$

In the following initial conditions of zero are considered.

Numerical results

In order to test the numerical solutions of system (5), a comparison with a fourth order Runge-Kutta method was implemented when $\alpha = 1$. The positive constants considered are $\mu = 0.8$, $F = 0.46$ and $\omega = 1$. For the discrete step and simulation time, $h = 0.01$ and $T = 200$ were considered for all of the simulations. The phase trajectory obtained with (5) and Runge-Kutta is shown in figure 1.

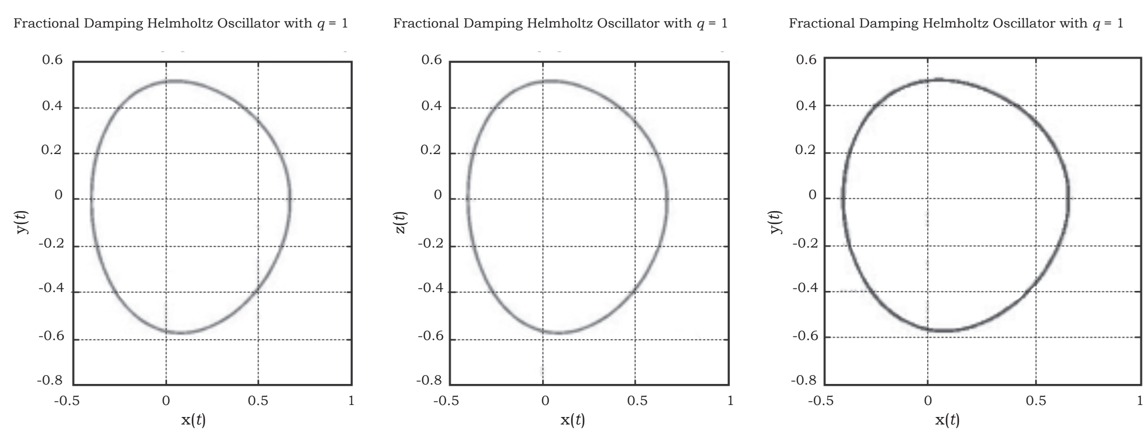

Figure 1. Phase trajectory comparison between fractional derivative (red) and Runge-Kutta method (blue).
Source: Authors own elaboration.

Effects of the Fractional Order Damping

In this work fractional order varies from 0.5 to 1.75 with the same parameters as the comparison with Runge-Kutta. Figure 2 shows cases with various α values. Periodic motions are found through simulations at certain intervals. The interval tested was α ε [0.5, 1.38]. In figure 2(a)-(b), α =0.5, 1, respectively, there is not much qualitative difference in the behavior of the system. Cyclic orbits and the oscillations of x versus t show that the main difference lies in the amplitude of such oscillations along the time of the computational experiment.

The behavior is very different when α =1.38, as shown in figure 2(c). Closed orbits remain, but the phase trajectory and $x(t)$ oscillations become irregular.

From α =1.39, with the same parameters and initial conditions of zero, the orbits escape. The escape criteria chosen in this work is: *if* $x(t_k)$>3 the trajectory escapes. When α =1.39 the orbit remains cyclic, but after t =51.23 it escapes (see figure 3(a)). For larger values of α, for example, α =1.75, the escape time of the orbit is shorter. Such case is shown in figure 3(b).

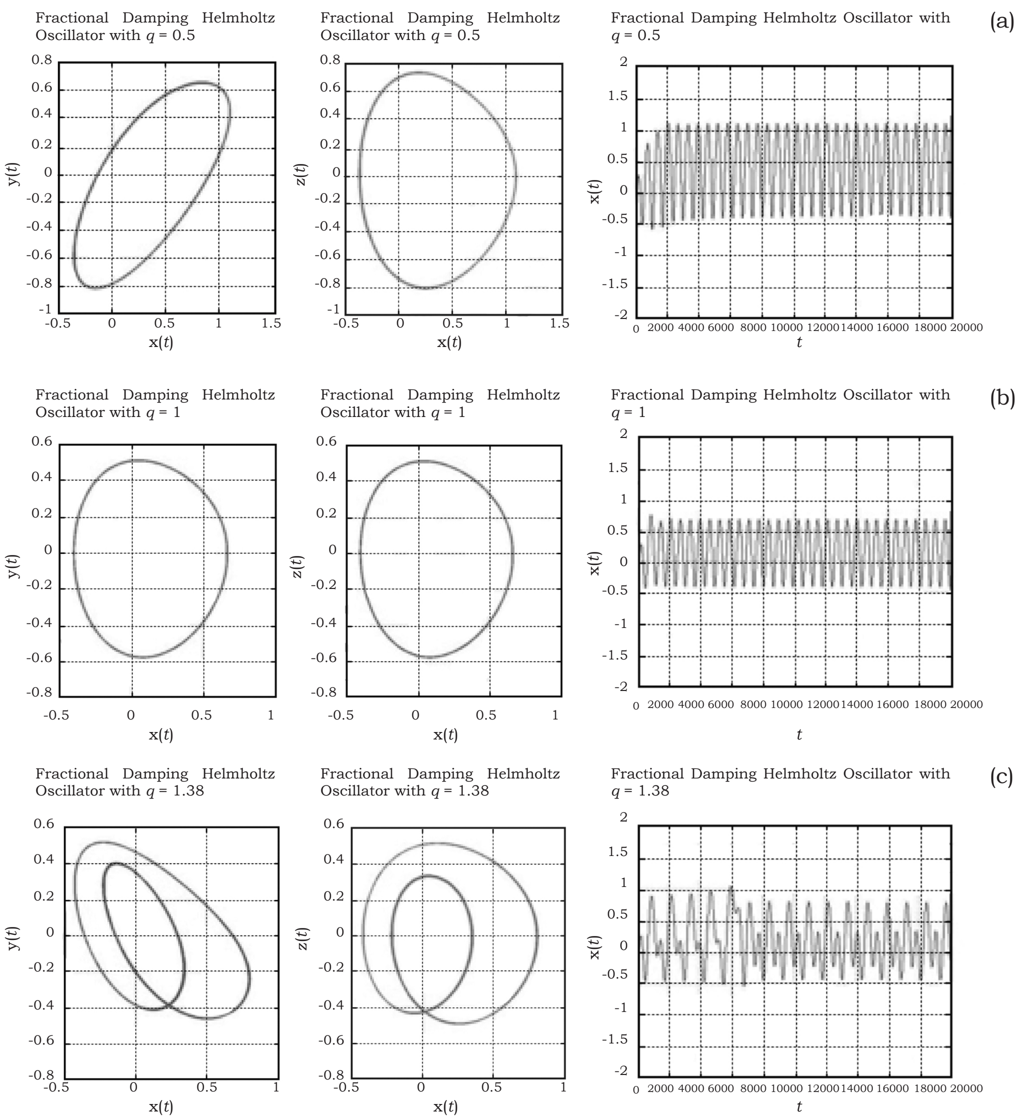

Figure 2. Phase trajectory with various α.
Source: Authors own elaboration.

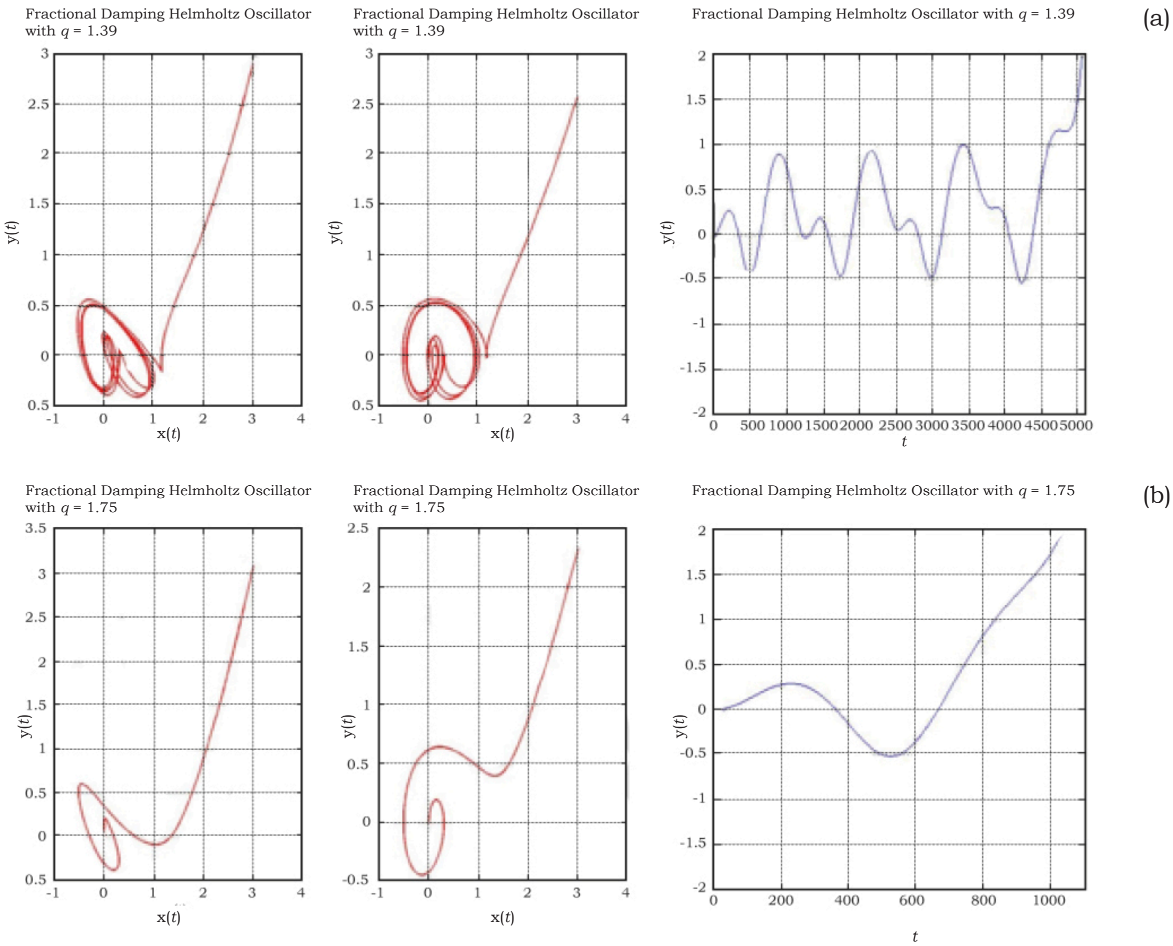

Figure 3. Escape of the trajectories at certain critical α.
Source: Authors own elaboration.

CONCLUSIONS

The nonlinear dynamics of the Helmholtz oscillator with fractional damping are studied in this work. The Grünwald-Letnikov fractional derivative is implemented numerically in order to get numerical simulations of the fractional behavior of the damping in the system. When α =1 the results are the same as the ones obtained with a fourth-order Runge-Kutta method.

In the simulations, the fractional damping exponent α is in the interval from 0.5 to 1.75 and two important behaviors were found. According the results, the α exponent works as a control parameter of the system. When $\alpha \leq 1.38$ the orbits are periodic and bounded and remain between the walls of the potential well. On the other hand, one can argue that α= 1.39 is a critical value of the control parameter, so that all the orbits escape. In particular, the bigger α the shorter the escape time from the well.

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